AoC

Zermelo's axiom of choice: let $\mathcal{F}$ be a nonempty collection of nonempty sets, then $\exists$ a choice fn.

we will return to orderings and Zorn's lemma in the future

Infinite cartesian product: Suppose you have $\mathbb{R}$, then its easy to construct $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n$

Does it make sense to talk about $\mathbb{R}^{\mathbb{R}}$?

DEFINE

$\mathbb{R}^{\mathbb{R}} := \{$ the set of all mappings from $\mathbb{R}$ to $\mathbb{R}\}$.

Or more generally, given an index set Λ and a collection $\{E_\lambda\}_{\lambda \in \Lambda}$

Define

$$\prod_{\lambda \in \Lambda} E_\lambda = \{ f : \Lambda \to \bigcup_{\lambda \in \Lambda} E_\lambda \mid f(\lambda) \in E_\lambda \ \forall \lambda \in \Lambda \}$$

What is the relationship between AoC and Cartesian Product

(not the democratic congresswoman from Queens)

Axiom of choice $\Leftrightarrow$ $\prod_{\lambda \in \Lambda} E_\lambda$ is not empty

This version is not satisfactory to me, since there may not exist an index set.

Choice fn: let X be a collection of nonempty sets. A choice fn f on X is st $\forall A \in X$, $f(A) \subset A$.

Axiom of Choice: $\forall$ collection X, $\exists$ a choice fn f.

2.6 lec 5 (Nonmeasurable sets on $\mathbb{R}$)

Lemma 16 Let E be bounded & meas. Suppose $\exists$ a bounded countably a set of real numbers Λ st $\{\lambda+E\}_{\lambda\in\Lambda}$ are disjoint.

Then $m(E)=0$.

Pf: $$m\Big(\overbrace{\bigcup_{\lambda\in\Lambda}\{\lambda+E\}}^{\text{bounded.}}\Big) = \sum_{\lambda\in\Lambda} m(\lambda+E) = \sum_{\lambda\in\Omega} m(E)$$

Since Λ is bounded & E is bounded the sum is finite an convergent. So we must have $m(E)=0$

Equivalence relation: $x\sim y$ if $x-y\in\mathbb{Q}$.
on $\mathbb{R}$.

You can define $\sim$ on any subset $E\subset\mathbb{R}$.

$E_x=\{y\in E : x-y\in\mathbb{Q}\}$ This is the equivalence class of x.

Two distinct equivalence classes are always disjoint. Easy to see

Let $z\in E_x\cap E_y$ and $x'\in E_x$. Then $x'\sim x\sim z\sim y$
$\Rightarrow x'\in E_y$

Let $X=\{$set of all equivalence classes of $E\}$

By AoC $\exists$ a choice fn on X let $C_E = f(X)$

The choice set $\mathcal{C}_E$ that consists of one member from each equivalence class in E

Thm: (Vitali) Every E of positive outer measure contains a nonmeas. subset.

Pf: $E = \bigsqcup_{n=-\infty}^{\infty} E \cap (n, n+1]$ $\quad \sum_n m^*(E \cap (n,n+1]) \geq m^*(E) > 0$

So we may assume E is bounded. $E \subseteq [-b, b]$

let $x \sim y$ if $x - y \in \mathbb{Q}$

let X be the set of equivalence classes on E and let $\mathcal{C}_E$ be a choice set (by AoC).

Then $\mathcal{C}_E$ satisfies an amazing property:

If $\Lambda \subseteq \mathbb{Q}$, then $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is disjoint

This is because if $x \in \lambda + \mathcal{C}_E$ and $y \in \lambda' + \mathcal{C}_E$

then if $x = y$ $\left(\lambda_1 + z_1 = \lambda_2 + z_2 \Rightarrow z_1 - z_2 = \lambda_2 - \lambda_1 \in \mathbb{Q} \right)$

This means $z_1 - z_2$ MUST be in the same equivalence class.

But this cannot be true since the choice fn picks out one from each equivalence class.

Then $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Omega}$ is disjoint. Choose $\Lambda = \mathbb{Q} \cap [-2b, 2b]$

Then both $E \subseteq [-b, b]$ $\Lambda \subseteq [-2b, 2b]$ are bounded,

and hence $m^*(\mathcal{C}_E) = 0$, by the previous lemma.

We must have $\bigcup_{\lambda \in \Lambda} \{\lambda + \mathcal{C}_E\} \supseteq E$

Let $x \in E$. Look at E_x and let y be its representative in $\mathcal{C}_E$.
Then $x-y \in \mathbb{Q}$ so let $\lambda = x-y$. Since $x,y \in [-b,b]$ $x-y \in [-2b,2b] \cap \mathbb{Q}$
$\Rightarrow x \in \lambda + \mathcal{C}_E$ for $\lambda \in \Lambda$. Then

$$m^*(E) \leq \sum_{\lambda \in \mathbb{Q}} m^*(\lambda + \mathcal{C}_E) = 0$$ (using translation invariance)

This contradicts $m^*(E) > 0$.

Thm: $\exists$ A, B disjoint st

$$m^*(A \sqcup B) < m^*(A) + m^*(B)$$

Pf: If not for any E, $A = A \cap E \sqcup A \cap E^c$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \forall \text{ set } E$$

So every set E is meas.

This is NOT true.

☆ Can we explicitly construct (using AoC) such A and B?